UCLA
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Economics 262P: Population Economics

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STATIC MODELS OF THE ALLOCATION OF TIME AND GOODS

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I. DIRECT UTILITY FUNCTION APPROACH

(a) **Conceptual model**
Assume there is full information, no savings and an individual consumer-worker maximizes a real-valued, continuous, twice-differentiable, quasi-concave utility function:

$$\max_{x,\ell} u = u(x,\ell;A,\varepsilon) \tag{1}$$

x = consumption
ℓ = time devoted to leisure
$= T - h$ (2)

T = total time endowment
h = time devoted to market work
A = <u>observable</u> characteristics of individual (age, sex)
ε = <u>unobservable</u> individual characteristics (tastes)

subject to budget constraint:

$px = c(h) + y$

y = non-labor income

Assume wages are fixed

$c(h) = wh$

then budget constraint is <u>linear</u>:

$$px = wh + y \tag{3}$$

$$\Rightarrow [px+w\ell] = wT + y \tag{4}$$

p = price of consumption good
w = wage rate
wT + y = "full income"

(b) **Lagrangian**

$$L = u(x,\ell;A,\varepsilon) + \lambda[wT+y-px-w\ell] \tag{5}$$

(c) Assume an **interior solution** for labor supply [h>0] and commodity demand [x>0].
Then <u>FONC</u> are

$$u_x = \lambda p \tag{6}$$
$$u_\ell = \lambda w \tag{7}$$
$$wT + y - w\ell - px = 0 \tag{8}$$

(d) **Second order conditions**:

For maximum, require $d^2u < 0$ -- i.e., Hessian negative definite s.t.

$$H = \begin{bmatrix} u_{xx} & u_{x\ell} \\ u_{\ell x} & u_{\ell\ell} \end{bmatrix} \text{ is negative definite} \qquad (9)$$

i.e. $z'Hz < 0 \quad \forall z$ satisfying (4).

Stronger requirement which is useful would be:

$$z'Hz < 0 \quad \forall z$$

Then (9) can be expressed as:

$$[dx \;\; d\ell \;\; d\lambda] \begin{bmatrix} u_{xx} & u_{x\ell} & -p \\ u_{\ell x} & u_{\ell\ell} & -w \\ -p & -w & 0 \end{bmatrix} \begin{bmatrix} dx \\ d\ell \\ d\lambda \end{bmatrix} \leq 0 \qquad (10)$$

NOTE: if H is negative definite then

$$u_{xx} < 0 \qquad (11)$$

$$u_{xx}u_{\ell\ell} - u_{x\ell}^2 > 0$$

$$\Rightarrow u_{\ell\ell} < 0 \qquad (12)$$

(e) **[Marshallian] labor supply and commodity demand functions:**

From (6) and (7)

$$\frac{w}{p} = \frac{u_\ell}{u_x} = m(x, \ell, y; \; A, \varepsilon)$$

and applying (8)

$$\underline{\ell = \ell(p, w, y; \; A, \varepsilon)} \qquad (14)$$

$$\underline{x = x(p, w, y; \; A, \varepsilon)} \qquad (15)$$

(f) **Properties of Marshallian demand functions:**

Totally differentiate (6) through (8)

$$\begin{bmatrix} u_{xx} & u_{x\ell} & -p \\ u_{\ell x} & u_{\ell\ell} & -w \\ -p & -w & 0 \end{bmatrix} \begin{bmatrix} dx \\ d\ell \\ d\lambda \end{bmatrix} = \begin{bmatrix} \lambda dp \\ \lambda dw \\ Xdp + \ell\, dw - dy - Tdw \end{bmatrix} \tag{16}$$

by inversion:

$$\begin{bmatrix} dx \\ d\ell \\ d\lambda \end{bmatrix} = B \begin{bmatrix} \lambda dp \\ \lambda dw \\ Xdp + \ell\, dw - dy - Tdw \end{bmatrix} \tag{17}$$

where B is the inverse of the first term in (16)

(i) <u>Effect of change in non-labor income, $dy \neq 0$</u>

$$\frac{\partial x}{\partial y} = -b_{13} \gtreqless 0 \tag{18}$$

$$\frac{\partial \ell}{\partial y} = -b_{23} \gtreqless 0 \tag{19}$$

(ii) <u>Effect of change in price</u>

(a) <u>Own price effect</u>

$$\frac{\partial x}{\partial p} = \lambda b_{11} + x b_{13}$$

From (18)

$$= \lambda b_{11} - x \frac{\partial x}{\partial y} \tag{20}$$

To interpret (20) consider how y, p, w would have to change in order to keep u constant:

$$0 = du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial \ell} d\ell \qquad \text{by chain rule}$$

substituting (6) and (7)

$$0 = du = \lambda\ [pdx + wd\ell]$$

but from budget constraint

$$pdx + wd\ell + xdp + \ell dw = dy + Tdw$$

$$\Rightarrow \quad [pdx + wd\ell] = dy - xdp + (T-\ell)dw$$

by substitution

$$\Rightarrow \quad 0 = du = \lambda[dy - xdp + (T-\ell)dw]$$

Now, since $\lambda > 0$

if $du = 0$ then $(T-\ell)dw - xdp + dy = 0$

Substituting in third row of vector on RHS of (17)

$$\begin{bmatrix} dx \\ d\ell \\ d\lambda \end{bmatrix} = B \begin{bmatrix} \lambda dp \\ \lambda dw \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \left.\frac{\partial x}{\partial p}\right|_{du=0} = \lambda b_{11}$$

Substituting in (20)

$$\frac{\partial x}{\partial p} = \left.\frac{\partial x}{\partial p}\right|_{du=0} - x\frac{\partial x}{\partial y} \qquad (21)$$

$\longrightarrow$ ... <u>Slutsky decomposition</u>

Effect of price change can be decomposed into a substitution effect (SE) and an income effect (IE):

$$\frac{\partial x}{\partial p} = SE + IE$$

Since B is negative definite, $b_{ii} < 0 \ \forall i$, and $\lambda b_{ii} < 0$. The pure substitution effect of a price change reduces the demand for x. If x is "normal", then

$$\frac{\partial x}{\partial y} > 0 \text{ and } \left[-x\frac{\partial x}{\partial y}\right] < 0.$$

The income effect reinforces the substitution effect.

Note that if x is "sufficiently" inferior $(\partial x/\partial y < 0)$ then $\partial x/\partial p$ could be positive and the demand curve would be upward sloping.

Labor supply case

You should be able to show that

$$\frac{\partial \ell}{\partial w} = \left.\frac{\partial \ell}{\partial w}\right|_{du=0} + (T-\ell)\,\frac{\partial \ell}{\partial y} \tag{22}$$

$$= SE + IE.$$

Again the substitution effect is negative. If the worker-consumer is a net <u>supplier</u> of labor, $T-\ell > 0$, and leisure is normal, then the income effect of an increase in the wage increases the demand for leisure. If this effect is strong enough then leisure demand may be upward-sloping.
Since:

$$h = T - \ell$$

effect of change in wage on hours of labor supply

$$\frac{\partial h}{\partial w} = -\frac{\partial \ell}{\partial w} = -\left.\frac{\partial \ell}{\partial w}\right|_{du=0} - h \bullet \frac{\partial \ell}{\partial y} \tag{23}$$

The substitution effect of a wage change on labor supply is clearly positive. If leisure is normal, the income effect is negative and hence the sign of $\partial h/\partial w$ is *a priori* ambiguous. The labor supply curve might be "backward bending."

(b) <u>Cross price effects</u>

$$\frac{\partial \ell}{\partial p} = \left.\frac{\partial \ell}{\partial p}\right|_{du=0} - x\,\frac{\partial \ell}{\partial y}$$

$$\frac{\partial x}{\partial w} = \left.\frac{\partial x}{\partial w}\right|_{du=0} + (T-\ell)\,\frac{\partial x}{\partial y}$$

Since B is symmetric (see (16) and (17)), the Slutsky matrix of compensated effects is also symmetric:

$$\left.\frac{\partial x}{\partial w}\right|_{du=0} = \left.\frac{\partial \ell}{\partial p}\right|_{du=0} \tag{24}$$

(g) **Aggregation conditions**

The non-satiation assumption implies consumers will always choose bundles which lie on the budget constraint:

$$px - (T-\ell)w = y$$

(i) Differentiating wrt income:

$$p\frac{\partial x}{\partial y} + w\frac{\partial \ell}{\partial y} = 1 \qquad \text{... Engel aggregation condition} \qquad (25)$$

$\Rightarrow$ price weighted marginal propensities to consume sum to unity.

In shares:

$$\frac{px}{y}\,\xi_{xy} + \frac{w\ell}{y}\,\xi_{\ell y} = 1$$

$\Rightarrow$ share weighted sum of income elasticities sum to unity.

(ii) Differentiating wrt p and w:

$$\begin{aligned} p\frac{\partial x}{\partial p} + w\frac{\partial \ell}{\partial p} + x &= 0 \\ p\frac{\partial x}{\partial w} + w\frac{\partial \ell}{\partial w} - (T-\ell) &= 0 \quad \text{... Cournot aggregation condition} \end{aligned} \qquad (26)$$

$\Rightarrow$ price weighted sum of responses to a change in the i^{th} price = - amount of i^{th} good consumed.

(h) **Homogeneity**

(i) Demands for x and ℓ are <u>homogeneous of degree zero</u> in (p,w,y).

From (6) and (7)

$$\frac{u_\ell}{u_x} = \frac{w}{p}$$

and budget constraint

$$y + w(T-\ell) - px = 0$$

which determine optimal values of ℓ and x. If p, w, and y are all multiplied by some factor k, then these conditions remain unchanged and therefore the optimal values of ℓ and x are unaffected. $\Rightarrow$ There is no money illusion.

(ii) In general $g(\underline{z})$ is <u>homogeneous of degree k</u> if

$$g(t\underline{z}) = t^k g(\underline{z})$$

differentiating wrt t

$$\Sigma_i g_i z_i = k t^{k-1} g(\underline{z}) \quad \text{where} \quad g_i = \frac{\partial g(\bullet)}{\partial z_i}$$

if $g(\bullet)$ is homogeneous of degree zero, $k = 0$ and $\Sigma_i g_i z_i = 0$.

Applying to demand functions:

$$x(tp, tw, ty) = t^k x(p,w,y) = x(p,w,y)$$

$$p\frac{\partial x}{\partial p} + w\frac{\partial x}{\partial w} + y\frac{\partial x}{\partial y} = 0$$

Substituting (21) and (22) and rearranging

$$p\frac{\partial x}{\partial p}\bigg|_{du=0} + w\frac{\partial x}{\partial w}\bigg|_{du=0} + \frac{\partial x}{\partial y}\{y-px+w(T-\ell)\} = 0$$

but $\{\ \} = 0$

$$\Rightarrow \qquad p\frac{\partial x}{\partial p}\bigg|_{du=0} + w\frac{\partial x}{\partial w}\bigg|_{du=0} = 0 \qquad (27)$$

$\Rightarrow$ <u>price-weighted sum of Slutsky terms (compensated price effects) is zero</u>

Note, of course

$$w\frac{\partial \ell}{\partial w}\bigg|_{du=0} + p\frac{\partial \ell}{\partial p}\bigg|_{du=0} = 0$$

(i) **2-good case**

Since $\left.\frac{\partial x}{\partial p}\right|_{du=0} < 0$ and $\left.\frac{\partial \ell}{\partial w}\right|_{du=0} < 0$, in the 2-good case it is clear that $\left.\frac{\partial x}{\partial w}\right|_{du=0} = \left.\frac{\partial \ell}{\partial p}\right|_{du=0}$ must be positive. Goods and leisure must be substitutes. In the more general n-good case this restriction does not apply.

(j) **General n-good Case**

$$\text{Max}_x \; u(x_1, \ldots, x_n)$$
$$\text{s.t.} \quad \Sigma_i p_i (x_i - \bar{x}_i) = y \qquad \bar{x} = \text{endowment}$$

(i) Slutsky breakdown

$$\frac{\partial x_i}{\partial p_j} = S_{ij} - (x_j - \bar{x}_j)\frac{\partial x_i}{\partial y} \qquad S \text{ symmetric and negative definite}$$

(ii) Homogeneity

$$\Sigma_j p_j \, \partial x_i/\partial p_j + y\partial x_i/\partial y = 0 \qquad \forall i$$
$$\Sigma_j p_j \, x_{ij} = 0 \qquad \forall i$$

(iii) Engel aggregation

$$\sum_i p_i \frac{\partial x_i}{\partial y} = 1$$

(iv) Cournot aggregation

$$\sum_k p_k \frac{\partial x_k}{\partial p_i} + x_i = 0$$

II. INDIRECT UTILITY FUNCTION APPROACH

The optimal consumption rules for goods and leisure in the face of known prices and nonlabor income are given by (14) and (15). Since they define the consumption levels that would be chosen if utility is maximized, the highest level of utility attainable is given by

$$V(p,w,y) = u[x^*(p,w,y), \ell^*(p,w,y)] \tag{28}$$

which is indirect utility function. It is an indication of how the welfare of a consumer-worker changes as prices, wages or nonlabor income change.

(a) **Effect of change in income**
Intuitively, an increase in nonlabor income should increase welfare.

$$\begin{aligned} \frac{\partial V}{\partial y} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial u}{\partial \ell}\frac{\partial \ell}{\partial y} \\ &= \lambda\left[p\frac{\partial x}{\partial y} + w\frac{\partial \ell}{\partial y}\right] && \text{from FONC (6) and (7)} \\ &= \lambda && \text{from Engel aggregation (25)} \qquad (29) \\ &> 0 && \text{since } \lambda = \frac{U_x}{p} > 0 \end{aligned}$$

λ = MU of nonlabor income.

(b) **Effect of change in price**

$$\begin{aligned} \frac{\partial V}{\partial p} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial u}{\partial \ell}\frac{\partial \ell}{\partial p} \\ &= \lambda\left[p\frac{\partial x}{\partial p} + w\frac{\partial \ell}{\partial p}\right] && \text{from FONC (6) and (7)} \\ &= -\lambda x && \text{from Cournot Aggregation (26)} \qquad (30) \\ &\le 0 \end{aligned}$$

Note if $x = 0$ then $\partial V/\partial p = 0$, i.e., a change in the price of a good not consumed has no effect on welfare.

(c) **Effect of change in wage**
Intuitively an increase in wage should have a non-negative effect on welfare.

$$\begin{aligned} \frac{\partial V}{\partial w} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial u}{\partial \ell}\frac{\partial \ell}{\partial w} \\ &= \lambda[T-\ell] && \text{FONC and Cournot aggregation} \\ &= \lambda h \ge 0 && (31) \end{aligned}$$

(d) **Roy's Identity**
From (29) and (30)

$$\frac{\partial V/\partial p}{\partial V/\partial y} = -x(p,w,y) \qquad \text{Marshallian consumer demand} \tag{32}$$

$$\begin{aligned} \frac{\partial V/\partial w}{\partial V/\partial y} &= h(p,w,y) \\ &= T - \ell(p,w,y) \end{aligned} \tag{33}$$

$\Rightarrow$ Given the indirect utility function, commodity demand and labor supply functions are easily derived.

III. COST FUNCTION/EXPENDITURE FUNCTION APPROACH

Dual to the utility maximization problem is the cost-minimization problem. The aim is to minimize total expenditure on commodities in excess of labor income ($px-wh = px- w(T-\ell)$) subject to achieving some minimum level of utility, u_o:

$$\min_{x,\ell} c(p,w,;A,\varepsilon) = px - w(T-\ell) \tag{34}$$

$$\text{s.t } u(x,\ell;A,\varepsilon) \geq u_0$$

Assuming the cost function is concave in prices, continuous and twice differentiable then:

FONC:

$$\begin{aligned} p &= \mu u_x \\ w &= \mu u_\ell \\ u_o &= u(x,\ell) \end{aligned} \tag{35}$$

which can be solved to give the Hicksian demand functions:

$$\begin{aligned} &\underline{x = x(p,w,u_o)} \\ &\underline{\ell = \ell(p,w,u_o)} \end{aligned} \tag{36}$$

Following the method used above (cf., Section I.f above), you should be able to derive the properties of these compensated demand functions and their relationship to Marshallian demand functions.

More directly: at the optimum, the (excess) expenditure function is simply the amount of nonlabor income required to achieve u_o, the highest attainable utility given prices, p, wages, w, and nonlabor income y.

$$\begin{aligned} y &\equiv c(p,w,u_o) \\ &\equiv c(p,w,u_o(p,w,y)) \end{aligned} \tag{37}$$

By the chain rule

$$1 = c_3 \bullet \frac{\partial u_o}{\partial y} \quad \text{where}$$

$$c_j = \frac{\partial c}{\partial \arg(j)} \tag{38}$$

from (29)

$$\frac{\partial u_o}{\partial y} = \lambda = \frac{1}{c_3} \tag{39}$$

Again by the chain rule

$$0 = c_1 + c_3 \bullet \frac{\partial u_o}{\partial p}$$

$$= c_1 + \frac{\partial u_o/\partial p}{\partial u_o/\partial y}$$

By Roy's identity

$$\frac{\partial u_o/\partial p}{\partial u_o/\partial y} = -x(p,w,u_o) \qquad (40)$$

$$\therefore\ c_1 = x(p,w,u_o) \qquad (41)$$

$\therefore$ Derivative of the cost function gives utility constant (compensated) demand function, and

$$c_2 = -h(p,w,u_o) = -[\,T - \ell(p,w,u_o)\,] \qquad (42)$$

This property is sometimes known as Shepherd's Lemma.

IV. HOMOTHETICITY

Preferences are said to be homothetic if for some normalization of u, doubling all quantities consumed results in doubling utility. Intuitively, indifference curves are magnified versions of one another.

Preferences are homothetic if

$$u(x) = \phi[v(x)] \qquad (43)$$

$v(x)$ is homogeneous of degree one
$\phi(\bullet)$ is monotone increasing.

The cost function is

$$c(u,p) = \min_x \{px;\ u = \phi[v(x)]\} \qquad (44)$$

$$= \min_x \{px;\ \phi^{-1}(u) = v(x)\}$$

Rescale x such that $x^* = x\phi(u)$

$$c(u,p) = \min_{x^*} \phi^{-1}(u)\ \{px^*;\ 1 = v(x^*)\}$$

Define

$$\gamma(u,p) = c(u,p)\phi(u)$$

$$\gamma(u,p) = \min_{x^*} \{px^*;\ 1 = v(x^*)\} \qquad (45)$$

which is independent of u.

Thus if preferences are homothetic then

$$c(u,p) = \phi^{-1}(u)b(p)$$

Notice also since

$$\frac{\partial \ln c(u,p)}{\partial \ln p_i} = \omega_i$$

where ω_i is the budget share devoted to good i

$$= \frac{\partial \ln b(p)}{\partial \ln p_i}$$

Budget shares are independent of utility and income; Engel curves are linear.

V. SEPARABILITY

(a) Weak Separability

Consider a set of demands $\underline{x}$ where $\underline{x} = \{\underline{x}_k\ \underline{x}_{\bar{k}}\}$. Let

$$u(\underline{x}) = f(\ v_k(\underline{x}_k)\ ,\ \underline{x}_{\bar{k}}\ ) \tag{46}$$

then if the ordering of $\underline{x}_k$ is independent of $\underline{x}_{\bar{k}}$ then $\underline{x}_k$ is <u>weakly separable</u> from $\underline{x}_{\bar{k}}$. $v_k(\underline{x}_k)$ is a <u>sub-utility</u> or <u>felicity</u> function.

Consider

$$u(\underline{x}) = f\{\ v_G(x_G)\quad v_H(x_H)\quad v_K(x_K)\ \} \tag{47}$$

where x_G, x_H and x_K are vector-valued. [If they are scalars then preferences over them are <u>completely separable</u>.] Consider two commodities, $i \in G$, $j \in H$, $G \neq H$. Then the ij^{th} element of the Slutsky matrix is

$$s_{ij} = \left.\frac{\partial x_i}{\partial Y_G}\frac{\partial Y_G}{\partial p_j}\right|_u = s_{ji} = \left.\frac{\partial x_j}{\partial Y_H}\frac{\partial Y_H}{\partial p_i}\right|_u \tag{48}$$

$$\Rightarrow \left.\frac{\frac{\partial Y_G}{\partial p_j}}{\frac{\partial x_j}{\partial Y_h}}\right|_u = \left.\frac{\frac{\partial Y_H}{\partial p_i}}{\frac{\partial x_i}{\partial Y_G}}\right|_u = \lambda_{GH} \tag{49}$$

and λ_{GH} is independent of i and j (the elements of group G and H goods respectively).

Using the first and third terms in (49) to express $\frac{\partial Y_G}{\partial p_j}$ in terms of an income effect

$$\frac{\partial Y_G}{\partial p_j} = \lambda_{GH}\frac{\partial x_j}{\partial Y_H}$$

substituting in (48) and applying the Chain Rule

$$s_{ij} = \lambda_{GH}\ \frac{\partial x_i}{\partial Y}\frac{\partial Y}{\partial Y_G}\frac{\partial x_j}{\partial Y}\frac{\partial Y}{\partial Y_H}$$

$$= \mu_{GH}\ \frac{\partial x_i}{\partial Y}\frac{\partial x_j}{\partial Y} \tag{50}$$

$$\mu_{GH} =:\ \lambda_{GH}\ \frac{\partial Y}{\partial Y_G}\frac{\partial Y}{\partial Y_H}$$

(50) is necessary and sufficient for <u>weak separability</u>. Intuitively, a restriction is placed on the substitutability between goods across commodity groups.

(b) **Strong Separability**

Preferences are said to be strongly separable if

$$u = F[v_1(\underline{x}^1)+v_2(\underline{x}^2 + \ldots + v_N(\underline{x}^N)] \tag{51}$$

If each element of the felicity functions, $v_1,\ldots,v_N$ is a scalar then preferences are additively separable.

Using (50) with $g \in G$, $h \in H$, $k \in K$

$$s_{gh} = \mu_{GH} \frac{\partial x_g}{\partial Y}\frac{\partial x_h}{\partial Y} \tag{52}$$

$$s_{gk} = \mu_{GK} \frac{\partial x_g}{\partial Y}\frac{\partial x_k}{\partial Y} \tag{53}$$

but the group $L = \{HK\}$ is also strongly separable from G.
Let $\ell \in L$, $\Rightarrow \ell \in H$ or $\ell \in K$

$$s^*_{g\ell} = \mu^*_{g\ell} \frac{\partial x_g}{\partial y}\frac{\partial x_h}{\partial y} = s_{gh} = \mu_{GH}\frac{\partial x_g}{\partial y}\frac{\partial x_h}{\partial y} \tag{54}$$

and

$$s^*_{g\ell} = \mu^*_{g\ell} \frac{\partial x_g}{\partial y}\frac{\partial x_k}{\partial y} = s_{gk} = \mu_{GK}\frac{\partial x_g}{\partial y}\frac{\partial x_k}{\partial y} \tag{55}$$

Divide (54) by (55)

$$\frac{\mu_{GH}\partial x_h/\partial y}{\mu_{GK}\partial x_k/\partial y} = \frac{\partial x_h/\partial y}{\partial x_k/\partial y}$$

$$\Rightarrow \mu_{GH} = \mu_{GK} = \mu_G \tag{56}$$

but reversing the above logic it must be that

$$\mu_{GH} = \mu_{GK} = \mu_H$$

and

$$\mu_{KG} = \mu_{KH} = \mu_K$$

$$\Rightarrow \mu_G = \mu_H = \mu_K = \mu$$

Thus:

$$s_{gh} = \mu \frac{\partial x_g}{\partial Y}\frac{\partial x_h}{\partial Y} \tag{57}$$